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PROBABILITY FORMULAS FOR DESCRIBING FRAGMENT SIZE DISTRIBUTIONS

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (hib) Mott's distribution has long been used to describe fragment populations which result from the detonation of shells filled with high explosive. This distribution is a special case of more general probability distributions in widespread use for far different applications. Still it can be given a rational basis by means of a derivation which employs appropriate ideas about randomly distributed defects in solids. Methods of grouping fragment populations are discussed and applications of a general probability distribution are made to a number of

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cases of natural fragmentation. In addition, an application is made to a case of partially controlled fragmentation, illustrating the versatility of this approach.

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I. MOTT'S DISTRIBUTION

During World War II in England, Mott and Linfoot[1] tried representing the number of fragments produced by the detonation of a shell or bomb by the distribution

$$dN = [N_T/3\mu)]\ (m/\mu)^{-2/3} \exp\ [-\ (m/\mu)^{1/3}]\ dm$$

$$= N_T \exp\ [-\ (m/\mu)^{1/3}]\ d\ [(m/\mu)^{1/3}] \qquad (1)$$

where dN is the number of fragments in the mass interval from m to m+dm. Here N_T is the total number of fragments as can easily be seen by letting $t=(m/\mu)^{1/3}$ and integrating Equation (1) from 0 to ∞, namely,

$$\int_0^\infty dN = N_T \int_0^\infty e^{-t}\ dt = N_T. \qquad (2)$$

The parameter μ can be related to the expected value of the distribution, $\bar{m}$, as follows:

$$\bar{m} = \int_0^\infty m\ (dN/N_T) = \int_0^\infty \mu t^3\ (e^{-t}dt) = \mu\ (3!). \qquad (3)$$

They used an experimental statistic, namely, the observed average mass of the fragments collected, as an estimator of the expected value $\bar{m}$ to obtain an estimate of the population parameter, $\mu = m_{AV}/3!)$, where $m_{AV} = M_T/N_T$ with M_T being the total fragment mass collected. This gave rough agreement with experimental data obtained for a 3.7 inch shell for example, at least for the smaller fragments. For the larger fragments (which often showed part of the original shell inner and outer walls as surfaces) they observed that a better description could be obtained by using 1/2 instead of 1/3 in Equation (1). This suggested to them a dominance of three-dimensional fracture in the production of small fragments and a dominance of two-dimensional fracture for larger fragments, leading to a different estimate for μ, namely, $\mu = m_{AV}/\ (2!)$. In applications Mott never actually split a population into two groups, using exponents 1/3 and 1/2, but used one or the other. He remarked that similar distribution laws had already been applied to the crushing of rock, and cited an

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1 N.F. Mott and E. H. Linfoot, "A Theory of Fragmentation," A. C. 3348, Jan 1943.

American reference[2].

In a later report, Mott[3] suggested that for the exponent 1/2 at least, the parameter μ could be estimated from the shell wall thickness, T, and its inner diameter, D, by the formula

$$\mu^{1/2} = C\, T^{5/6}\, D^{1/3}\, (1 + T/D) \qquad (4)$$

where the empirical constant C is larger for explosives which impart a lower launch velocity to the fragments. He did not give a similar formula for $\mu^{1/3}$, the form found for this parameter in Equation (1). Since real shells have variable T and D and C along their length, it is clear that Mott was here considering the idealized case of a right circular cylinder. Later[4] be considered this idealized case of a cylinder formed from stacked metal rings in more detail. Here fracture perpendicular to the shell axis is pre-determined, and only ring fracture need be considered. This is similar to the natural fragmentation of a real warhead if we look on the shell as made up of stacked rings of different size and shape.

Shortly after, a co-worker of Mott named Ursell[5] suggested that the one-dimensional fracture of a rod ought to be given by a Poisson distribution, and might be related to warhead fragmentation. Still later in the United States, Thomas[6] pointed out formulas of the type $\exp\,[-(m/\mu)^{1/\ell}]$ where ℓ = 1, 2 or 3 for one-, two- or three-dimensional fracture are merely probability distributions more or less suited to describing particular fragment populations. It is not very helpful to require a physical model which envisions simultaneous formation of fragments by means of planes traversing volumes, lines traversing planes or points dividing lines. More realistically, all of these

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2 Lienau, J. Franklin Inst. p485 (1935).

3 N. F. Mott, Fragmentation of H. E. Shells; a Theoretical Formula for the Distribution of Weights of Fragments, A.C. 3642, 1943.

4 N. F. Mott, "Fragmentation of Shell Cases," Proc. Roy. Soc. (Lond.) 189, 300 (1947).

5 H. D. Ursell, "Fragmentation Data and Theories of Fragmentation," A. C. 3817, 1943.

6 L. C. Thomas, "Comments on Mott's Theory of the Fragmentation of Shells and Bombs," BRL R398, Sept 1943. (AD #36152)

processes might be going on simultaneously and sequentially, with smaller fragments being formed later from larger fragments already formed. In spite of this, Thomas used $\ell = 2$ in his own applications to U. S. munitions, possibly because of the calculational difficulties involved with the use of non-integer ℓ at the time. Thomas seems to have been the first to describe the fragmentation of real warheads by applying Mott's formula to individual rings of different wall thickness and diameter formed by mentally slicing a shell perpendicular to its axis[7]. This practice is still in use today[8].

G. I. Taylor also considered the explosive fragmentation of metal rings with radial cracks starting on the outside of the shell case and propagating inward under the combined influence of tensile and compressive forces[9]. He also pointed out that cracks should propagate at about 45° to the circumferential and radial directions, further complicating fragment size and shape distributions. More complete treatments such as that of Nadai[10] indicate that a system of logarithmic spiral cracks should develop in the wall in the simple case of infinitely long cylinders of uniform wall thickness uniformly stressed from the interior. Of course, for real warheads the crack systems will be much more complicated. A further complicating factor is the existence of shock waves which reverberate in the shell wall as it expands[11].

In summary, the physics of real warhead fragmentation is so complicated that simplified models are not likely to be much help in predicting fragment size distributions. When confronted with problems of extreme complexity, physicists generally invoke some form of probability description as in statistical mechanics. In what follows we will try to point out that Mott's general procedure can be given a rational basis in statistical theory and can be improved somewhat

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7 L. H. Thomas, "Analysis of the Distribution in Mass, in Speed, and Direction of Motion, of the Fragments of the M71 (90mm) A. A. Shell, when Filled with TNT and when Filled with Ednatol," BRL R434, Dec 1943.

8 Glenn Randers-Pehrson, R. R. Karpp, C. E. Anderson, Jr. and H. J. Blische, "Shortfrag Users Guide," ARBRL-MR-03007, Mar 1980.

9 G. I. Taylor, "The Fragmentation of Tubular Bombs," in The Scientific Papers of Sir Geoffrey Ingram Taylor, Ed. by G. K. Batchelor (Cambridge: The University Press, 1963) v.3, p387.

10A. Nadai, Theory of Flow and Fracture of Solids, (N.Y.: McGraw-Hill, 1950) p539.

11F. E. Allison and J. T. Schriempf, "Explosively Loaded Metallic Cylinders, II," J. Appl. Phys. 31, 846 (1960).

once it is realized that we are using a general theory of random breakup rather than particular physical models. We will also point out the generality of the description by showing that it can be applied to controlled as well as to natural fragmentation. We will not attempt to link microscopic fracture theories to continuum fracture mechanics by constructing distribution functions which represent particular types of flaw activation rates as others have done[12]. Instead, we will postulate an overall defect activation rate and pursue the statistical consequences of such an assumption. In particular, we will inquire how general and simple we can keep the form of our distribution function without losing its ability to represent experimental data well enough for practical purposes.

II. A GENERAL DISTRIBUTION

A. Derivation of the Distribution.

Consider a solid body of volume y and any shape. Real solids generally contain many kinds of defects which can act as weak points when a stress is applied and serve to initiate cracks and fractures. Let us mentally divide this body into k elementary volumes of size $\varepsilon = y/k$, choosing k large enough so that on average each elementary volume ε contains one defect or incipient break. Let r be the average volume rate of defect activation under stress so that $r\varepsilon = ry/k$ is the probability of finding at least one such activated defect in ε. Then the probability of finding no such defects in ε is $1-r\varepsilon$. Since the distribution of defects and their activation under stress can be considered random events, the probability of observing exactly s such activations in k trials is given by the binomial distribution

$$B(s/k,ry) = \frac{k!}{s!(k-s)!} \left(\frac{ry}{k}\right)^s \left(1-\frac{ry}{k}\right)^{k-s} \qquad (5)$$

for $s = 0,1,2...k$ with $B = 0$ otherwise. If s=k for example, then all defects would be changed into breaks and the body would be subdivided as much as it could be by the defect mechanism. Since the number of defects in real solids of interest is very large, it should be adequate to consider the limit as $k \to \infty$ in such a way that the number of breaks or activated defects, ry, is large but far from infinite.

12D. R. Curran, L. Seaman and D. A. Shockey, "Dynamic Failure in Solids," Physics Today, Jan 1977, p. 46.

In particular

$$ry/k << ry << k \tag{6}$$

is the condition of interest. In this case from Equation (5)

$$\lim_{k\to\infty} B = \frac{(ry)^s}{s!}\left[\lim_{k\to\infty} \frac{k!}{(k-s)!k^s}\right]\left[\lim_{k\to\infty}\left(1-\frac{ry}{k}\right)^k\right]\left[\lim_{k\to\infty}\left(1-\frac{ry}{k}\right)^{-s}\right]$$

$$= \frac{(ry^s}{s!}\ [1]\left[e^{-ry}\right]\ [1] = p(s/ry) \tag{7}$$

where p(s/ry) is the Poisson distribution with parameter ry. The first limit in Equation (7) is seen to be unity if we divide numerator and denominator by (k-s)!, so

$$\lim_{k\to\infty} \frac{k(k-1)(k-2)\ldots(k-s+1)}{k^s} = \lim_{k\to\infty} (1)\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\ldots\left(1-\frac{(s-1)}{k}\right) = 1. \tag{8}$$

The third limit in Equation (7) is obviously unity. If we let u = ry and z = -u/k, then we see that the second limit in Equation (7) is just the (-u) power of the limit which defines the base of the natural logarithm, namely

$$\lim_{z\to o}\left[(1+z)^{1/z}\right]^{-u} = e^{-u} \quad . \tag{9}$$

Of course the Poisson distribution meets the requirement that

$$\sum_{s=o}^{\infty} p(s/u) = \sum_{s=o}^{\infty} e^{-u}\ \frac{u^s}{s!} = e^{-u}\sum_{s=o}^{\infty}\ \frac{u^s}{s!} = e^{-u}\ e^{u} = 1. \tag{10}$$

Let F(u) be the probability that at least one break (activated defect) will occur under a given stress. Then 1-F(u) is the probability that no break will occur. From Equation (7), the probability of no breaks (s=o) is 1-F(u) = e^{-u}, or

$$F(u) = 1-e^{-u} = \sum_{s=o}^{\infty} p(s/u) - e^{-u} = \sum_{s=1}^{\infty}\ \frac{u^s}{s!}\ e^{-u} \tag{11}$$

which exhibits the meaning of F(u) as the probability of observing at least one (one or more) break. We note in Equation (11) that F(u) has been written both as a continuous function of the variable u and as a sum over a discrete frequency distribution in which u plays the role of a parameter. This is a particular case of a more general relation as we shall see below. Of course F(u) can also be written as an integral over a continuous frequency distribution as follows:

$$F(u) = 1 - e^{-u} = \int_0^u e^{-t}\, dt = \int_0^u f(t)dt \qquad (12)$$

where

$$dF = e^{-t}\, dt = f(t)dt. \qquad (13)$$

Here f(t) is a frequency distribution and F(u) is a cumulative distribution, while 1- F(u) is a complementary cumulative distribution. Equations (11) and (12) are both particular cases of more general relations, namely,

$$F(u,c) = \sum_{s=c}^{\infty} \frac{u^s}{s!} e^{-u} = P(2u/2c) = \frac{\gamma(c,u)}{\Gamma(c)} = \int_0^u \frac{1}{\Gamma(c)}\, t^{c-1}\, e^{-t}\, dt. \qquad (14)$$

Equations (11) and (12) are obtained for the case c=1 in Equation (14) since the complete gamma function of unity is unity, $\Gamma(1) = 0! = 1$. The complementary form of Equation (14) is

$$1-F(u,c) = \sum_{s=0}^{c-1} \frac{u^s}{s!} e^{-u} = Q(2u/2c) = \frac{\Gamma(c,u)}{\Gamma(c)} = \int_u^{\infty} \frac{1}{\Gamma(c)}\, t^{c-1}\, e^{-t} dt \qquad (15)$$

and is discussed, for example, in the Bureau of Standards Handbook of Mathematical Functions[13] where tables of these functions are also given. Equation (14) relates partial sums over the Poisson distribution to the chi-square distribution, p(2u/2c) with

$$2u = \chi^2 = \sum_{\ell=1}^{2c} \left(\frac{\chi_\ell - \chi_o}{\sigma} \right)^2 \qquad (16)$$

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13 Handbook of Mathematical Functions, NBS Applied Mathematics Series, number 55, Nov. 1964, Ed. by M. Abramowitz and I. A. Stegun, section 26, especially 26.4.19, 26.4.21 and 26.4.2.

where 2c is called the number of degrees of freedom. As we see, these sums are also related to the incomplete and complete gamma functions, $\gamma(c,u)$ and $\Gamma(c)$. The integrand in Equation (14) is the gamma frequency distribution with scale factor unity and is a special case of the Pearson type III distribution:

$$P_{III} = \frac{1}{\beta\Gamma(c)} \; t^{c-1} \; e^{-t} \tag{17}$$

where

$$t = \left[(y - y_o)/\beta\right] \tag{18}$$

for $y_o < y < \infty$ and scale factor β [NBS Handbook, p930]. An even more general form of Equation (18) is

$$t = \left[(y - y_o)/\beta\right]^{1/\ell} \tag{19}$$

where $0<\ell<\infty$. This gives us Weibull's frequency distribution

$$dF = W(t)\,dt = e^{-t}\,dt = \exp\left[-\left(\frac{y - y_o}{\beta}\right)\frac{1}{\ell}\right] d\left[\left(\frac{y - y_o}{\beta}\right)\frac{1}{\ell}\right]$$

$$= \frac{1}{\beta\ell}\left(\frac{y - y_o}{\beta}\right)^{\frac{1}{\ell} - 1} \exp\left[-\left(\frac{y - y_o}{\beta}\right)^{\frac{1}{\ell}}\right] dy = W(y)\,dy \tag{20}$$

which contains the exponent $1/\ell$ as well as the scale factor β and the cutoff value y_o for a random variable y over the range $y_o<y<\infty$. Weibull[14] has applied this distribution to a great variety of phenomena, social as well as physical, chemical and biological, as have others after him. Gnedenko[15] had shown previously that Equation (20)

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14 W. Weibull, "A Statistical Distribution Function of Wide Applicability," J. Appl. Mech., Sep 1951, p293.

15 B. V. Gnedenko, "Limit Theorems for the Maximum Term of a Variational Series," Doklady Akad. Nank, USSR32, 1941.

is the third asymptotic distribution of smallest values[16]. We note that Mott's distribution in Equation (1) is a particular case of Equation (20) for $\ell = 3$, $y = m$, $y_o = m_o = o$, $\beta = \mu$ and $d(N/T_T) = W(t)dt = dF$. Mott's formula with $\ell \geq 2$ is another special case of an integer ℓ value. Since Equation (20) is a particular case of Equation (14) with $c = 1$ and t given by Equation (19), so is Mott's formula. A more general form of Weibull's distribution which may be used in Equation (14) with any allowable c-value is the general gamma distribution:

$$
\begin{aligned}
g(t)\,dt &= \frac{1}{\Gamma(c)}\, t^{c-1}\, e^{-t} dt \\
&= \frac{1}{\Gamma(c)} \left(\frac{y - y_o}{\beta}\right)^{\frac{c-1}{\ell}} \exp\left[-\left(\frac{y - y_o}{\beta}\right)^{1/\ell}\right] d\left[\left(\frac{y - y_o}{\beta}\right)^{1/\ell}\right] \\
&= \frac{1}{\beta \ell \Gamma(c)} \left(\frac{y - y_o}{\beta}\right)^{\frac{c}{\ell} - 1} \exp\left[-\left(\frac{y - y_o}{\beta}\right)^{\frac{1}{\ell}}\right] dy \\
&= g(y)\, dy \qquad (21)
\end{aligned}
$$

Here y is a general random variable, but in fragmentation applications we take it to be a volume, $y = m/\rho$, where ρ is the density of the shell case and m is mass. With the scale factor $\beta = \mu/\rho$, $c = 1$ and $\ell = 2$ or 3, Equation (21) gives Mott's formulas. Since $c = 1$ in Mott's formulas, we see that he is calculating the probability of at least one break occurring. We have also seen that it is a special form of general probability distributions and can be applied to many things besides fragmentation. As we mentioned above, Thomas pointed out that Mott's formulas are not necessarily connected to any simple model of the fragmentation process. More generally, we now see that they are not necessarily limited to fragmentation at all.

Let us return to Equation (14) and display some particular examples by way of illustration. If the parameter $u = 1$, the Poisson frequency function is $p = e^{-1}/(s!)$, which is plotted in Figure 1. In this case the chance of obtaining one break ($s = 1$) is

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16 E. J. Gumbel, "Statistical Theory of Extreme Values (Main Results)," c.6 in. in Contributions to Order Statistics, Ed. by A. E. Sarken and B. G. Greenberg (N.Y.): John Wiley and Sons, Inc, 1962.

the same as the chance of obtaining no breaks (s=0), an instance of the double mode or maximum of the Poisson formula. As is well known, the expected value of s is equal to the parameter u (as is the variance) for the Poisson distribution. That is, in Figure 1, $\bar{s} = u = 1$. Since the chance of obtaining no breaks is large, namely, $e^{-1} = 0.368$, the chance of obtaining at least one break (one or more breaks) is only 0.632. The chances of obtaining s = 2,3,4.... breaks become smaller as s increases and the chance of obtaining 5 or more breaks is quite small. From Equation (14) it is

$$F(1,5) = \sum_{s=5}^{\infty} e^{-1}/(s!) = 1 - \sum_{s=0}^{4} e^{-1}/(s!) = 1 - e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right]$$

$$= \gamma(5,1)/\Gamma(5) = \int_0^1 \frac{1}{4!}\, t^4\, e^{-t}\, dt = \frac{.08772}{24} = .00366 . \qquad (22)$$

If the parameter u is larger, say u = 5, the Poisson frequency is $e^{-5}\, 5^s/(s!)$ which is plotted in Figure 2. Now the chance of obtaining no breaks at all is quite small, namely $e^{-5} = 0.0067$, but is not zero. The expected number of breaks is $\bar{s} = u = 5$ while the probability of obtaining at least one break is now much larger, namely, from Equation (14)

$$F(5,1) = \sum_{s=1}^{\infty} \frac{5^s}{s!}\, e^{-5} = \gamma(1,5) = \int_0^5 e^{-t}\, dt = 1 - e^{-5} = 0.9933 \qquad (23)$$

while the probability of obtaining at least the expected number of breaks is

$$F(5,5) = \sum_{s=5}^{\infty} \frac{5^s}{s!}\, e^{-5} = \frac{\gamma(5,5)}{\Gamma(5)} = \frac{1}{4!}\int_0^5 t^4\, e^{-t}\, dt = 0.5595 . \qquad (24)$$

In short, an increase in the parameter u reduces the chance of no breaks at all and shifts the distribution to the right, since the expected number of breaks is always $\bar{s} = u$. A decrease in u has the opposite effect. If u<1, $\bar{s}$ is fractional, and the probability of obtaining no breaks at all is the mode or most likely event.

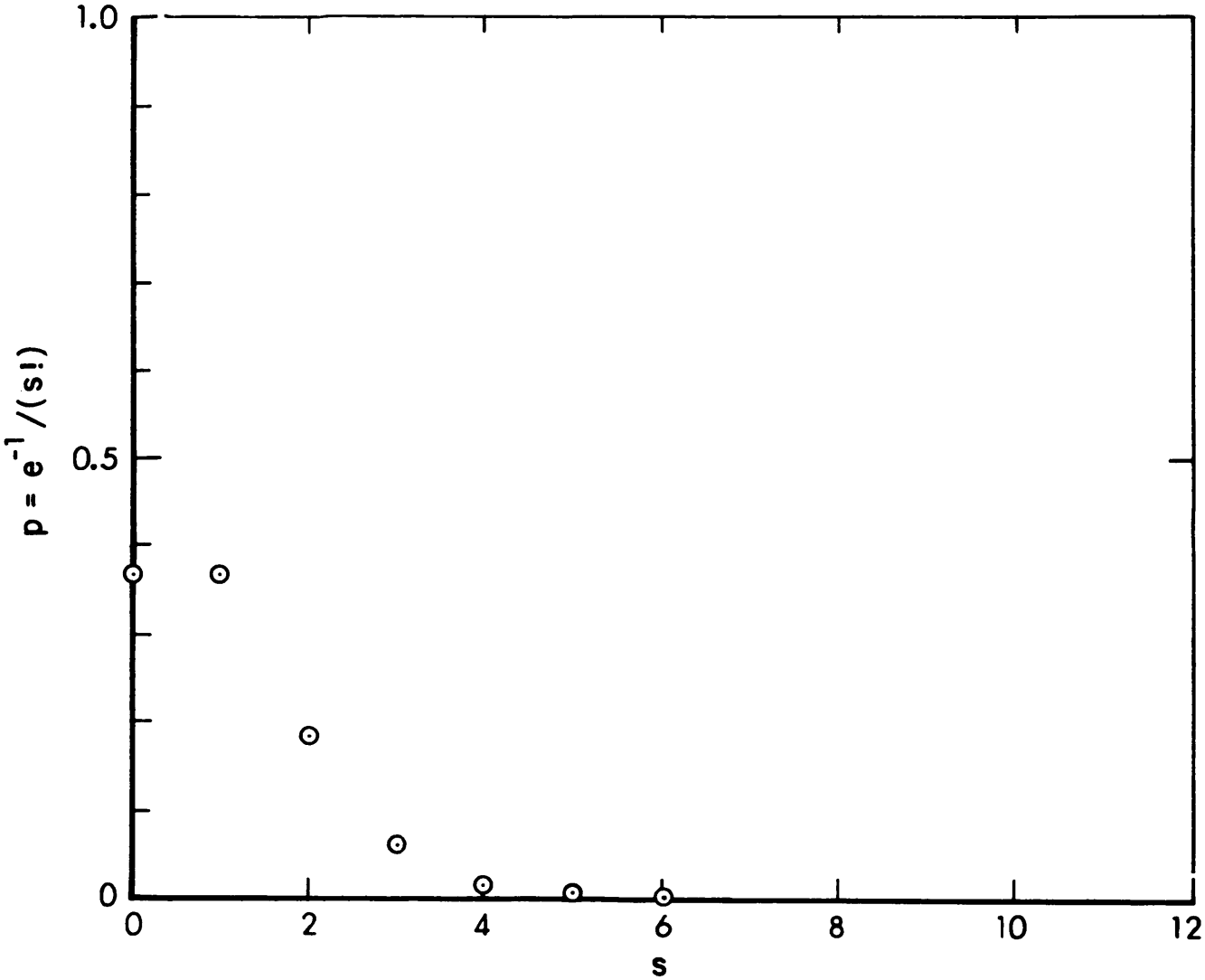

Figure 1. Poisson Frequency Distribution with Parameter $\mu = 1$

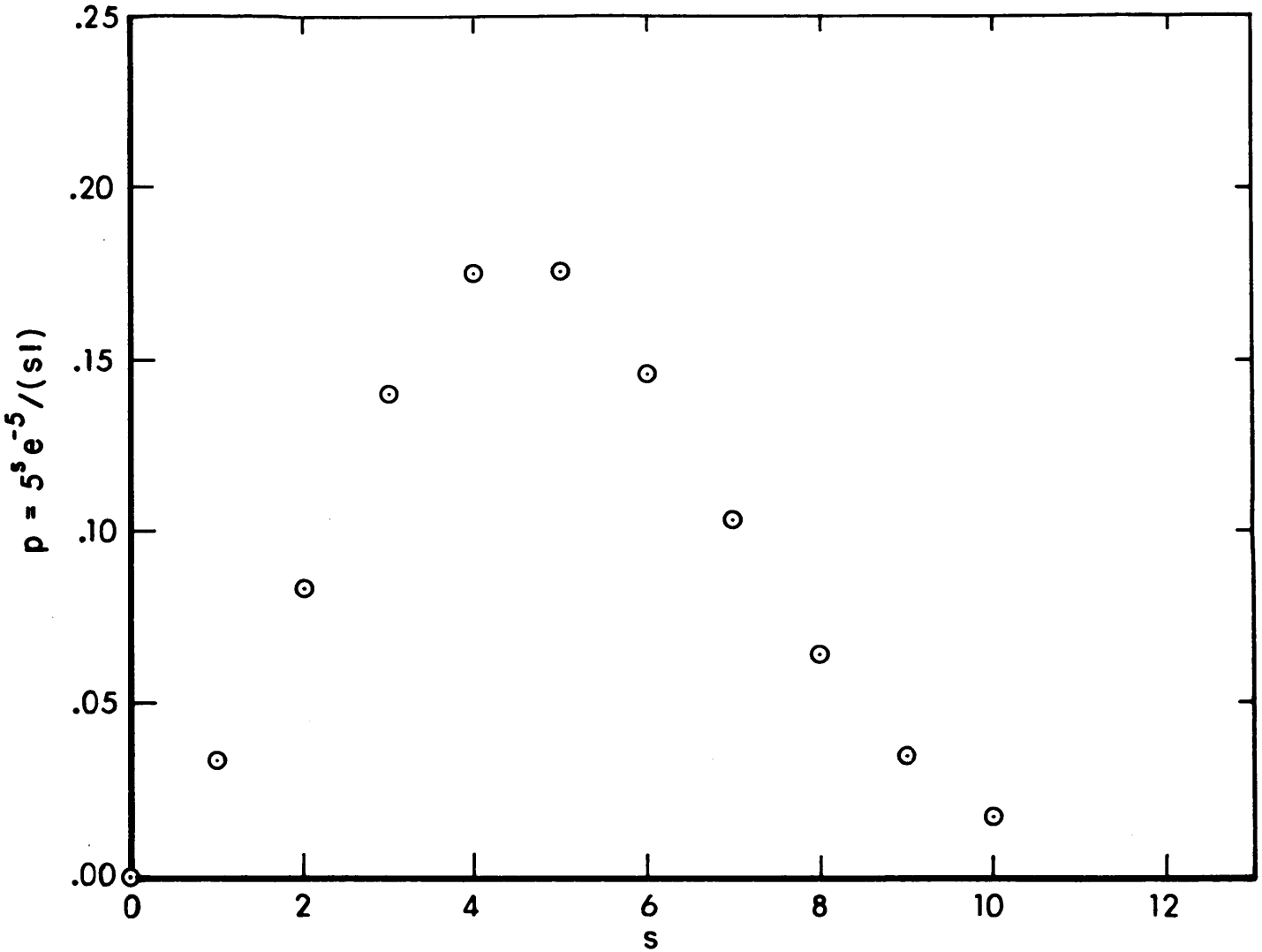

Figure 2. Poisson Frequency Distribution with Parameter $\mu = 5$.

In the limit as $u \to o$, the Poisson distribution for $s = o$ is a spike function equal to unity at $s = o$ and zero for all other s. Of course, $F(0,1) = 1-e^{o} = 0$, that is, the chance of at least one break vanishes. In other words for a perfect crystal for which $u = ry = 0$ because the rate of occurrence of defects $r = 0$, we are sure that no break will occur by a defect mechanism. In the limit as $u \to \infty$, the poisson function $(u^{s}/s!)/e^{u}$ is indeterminate. However, using L' Hospital's rule, s differentiations of the numerator reduces it to $(s!)/(s!)$ or unity, while the denominator remains the same, so that each term vanishes. Actually, u is not allowed to increase without limit as was pointed out in Equation (6) where $u = ry << k$. Even though k is allowed to increase without limit, u must remain finite. This requirement agrees with our interpretation of the dimensionless number u as the finite number of defects expected to be activated under a given stress. It also agrees with an interpretation of u as the ratio of the stress energy applied per unit volume to the work per unit volume required to fracture the body, namely,

$$u = \frac{\text{applied energy/volume}}{\sigma\varepsilon} \tag{25}$$

where σ and ε are the stress and strain at fracture characteristic of the material. In the case of projectiles striking a target at ordnance speeds, u will not be much greater than unity and the projectile will fracture into several pieces[17]. In this case discrete Poisson statistics are appropriate. For hypervelocity impacts, a projectile will shatter into many fragments, as also happens when explosive warhead cases are shattered at detonation. In such cases a continuous frequency distribution is more convenient for describing the resulting fragment population. Equation (14) shows how these distributions are connected. When $u = 1$ in Equation (25), the applied force is matched by the strength of the body and on average we expect only one break to occur, although it is equally likely that the body remain intact. When u is much less than unity we are in a regime of slow crack development and failure which may take months or years of stressing. For explosive ordnance we expect u much greater than one with fragmentation times in the millisecond range. But in all cases u will be finite and the upper limit of the integral in Equation (14) will not be infinite, although in some cases of interest it may be effectively infinite to a sufficient approximation.

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17 J. Dehn, "The Particle Dynamics of Target Penetration," ARBRL-TR-02188, Sept 1979.

If we use $y = m/\rho$ and $\beta = \mu/\rho$ in Equation (19) we obtain

$$t = \left(\frac{m - m_o}{\mu}\right)^{1/\ell} \quad . \tag{26}$$

and

$$u = t_{max} = \left(\frac{m_u - m_o}{\mu}\right)^{1/\ell} \tag{27}$$

so

$$m_o < m < m_u = m_o + \mu u^{\ell} \tag{28}$$

where m_u is the upper mass limit of a finite body. For u greater than ten, we see that the probability of at least one break occurring, $F(u) = 1-e^{u}$ is very close to unity. Since the number of fragments expected is one larger than the number of breaks, they are approximately equal for a large number of breaks.

For $u = 1$ in Equation (27), $\mu = (m_u - m_o)$ for any ℓ. In general, for $u \gtrless 1$, $\mu \lessgtr (m_u - m_o)$ for any ℓ $(0<\ell<\infty)$. Here μ is relatable to an expected average mass and smaller μ is associated with larger u, that is with greater applied stress, a weaker body and more breaks or fragments expected. The terms for small s in Equation (14) will be very small for large u, that is, the occurrence of only a few breaks is very unlikely. Most of the contribution to either the summation or integration in Equation (14) will come from the midrange near $s = u$ or m near μ. For example in Equation (14) we can take $c = 0$ or 1 and obtain the expected number of breaks

$$\bar{s} = \sum_{s=o}^{\infty} s\left(e^{-u}\frac{u^s}{s!}\right) = \sum_{s=1}^{\infty} s\left(e^{-u}\frac{u^s}{s!}\right) = ue^{-u}\sum_{j=o}^{\infty}\frac{u^j}{j!} = ue^{-u}e^{u} = u \tag{29}$$

since the term for $s = 0$ makes no contribution. Similarly, using $m = m_o + \mu t^{\ell}$ from Equation (26) we can find the expected values of the mass (letting $c = 1$)

$$\bar{m} = \frac{1}{\Gamma(1)} \int_o^u \left(m_o + \mu t^{\ell} \right) e^{-t} \, dt = m_o \, \gamma \, (1,u) + \mu \, \gamma \, (1+\ell,u) \qquad (30)$$

If u is effectively infinite, then the incomplete gamma function, γ, is approximately equal to the complete gamma function, Γ, and Equation (30) becomes

$$\bar{m} = m_o \, \Gamma(1) + \mu \, \Gamma \, (1+\ell) = m_o + \mu \, \Gamma \, (1+\ell) \, . \qquad (31)$$

If, in addition, the lower mass limit, m_o, is effectively zero and ℓ is an integer, Equation (31) becomes

$$\bar{m} = \mu \, (\ell!) \qquad (32)$$

since $\Gamma \, (1+\ell) = \ell!$ for integer ℓ. Mott's formulas consider only the values $\ell = 2$ or 3. In general, ℓ need not be integer, m_o need not be zero and m_u (and so u) is less than infinite.

In Figure 3 we plot the incomplete gamma function $\gamma(1+\ell,u)$ versus u for ℓ=0,1,2,3 and 4. For non-integer ℓ, the curves lie between those shown. It is clear that for smaller ℓ values the approximation $\gamma(1+\ell,u) \approx \Gamma(1+\ell)$ is quite good for smaller u than for larger ℓ values. For $0<\ell<3$, it is very good for $u \geq 10$. This was mentioned above in another way when we observed that $F(u)=1-e^{-u} \approx 1$ for $u \geq 10$.

B. Description of Fragment Populations

It is worthwhile noting the effects that various groupings can have on a given collection of fragments and how this can influence our mathematical representation. In Table I we present a sample population

Table I. SAMPLE FRAGMENT POPULATION (Masses in Grams)

1	.106	8	.264	15	1.250	22	3.950
2	.110	9	.268	16	1.411	23	4.922
3	.115	10	.311	17	1.706	24	5.700
4	.123	11	.450	18	1.972	25	5.850
5	.151	12	.525	19	2.002	26	7.106
6	.172	13	.713	20	2.150	27	9.760
7	.195	14	.809	21	3.670	28	10.500

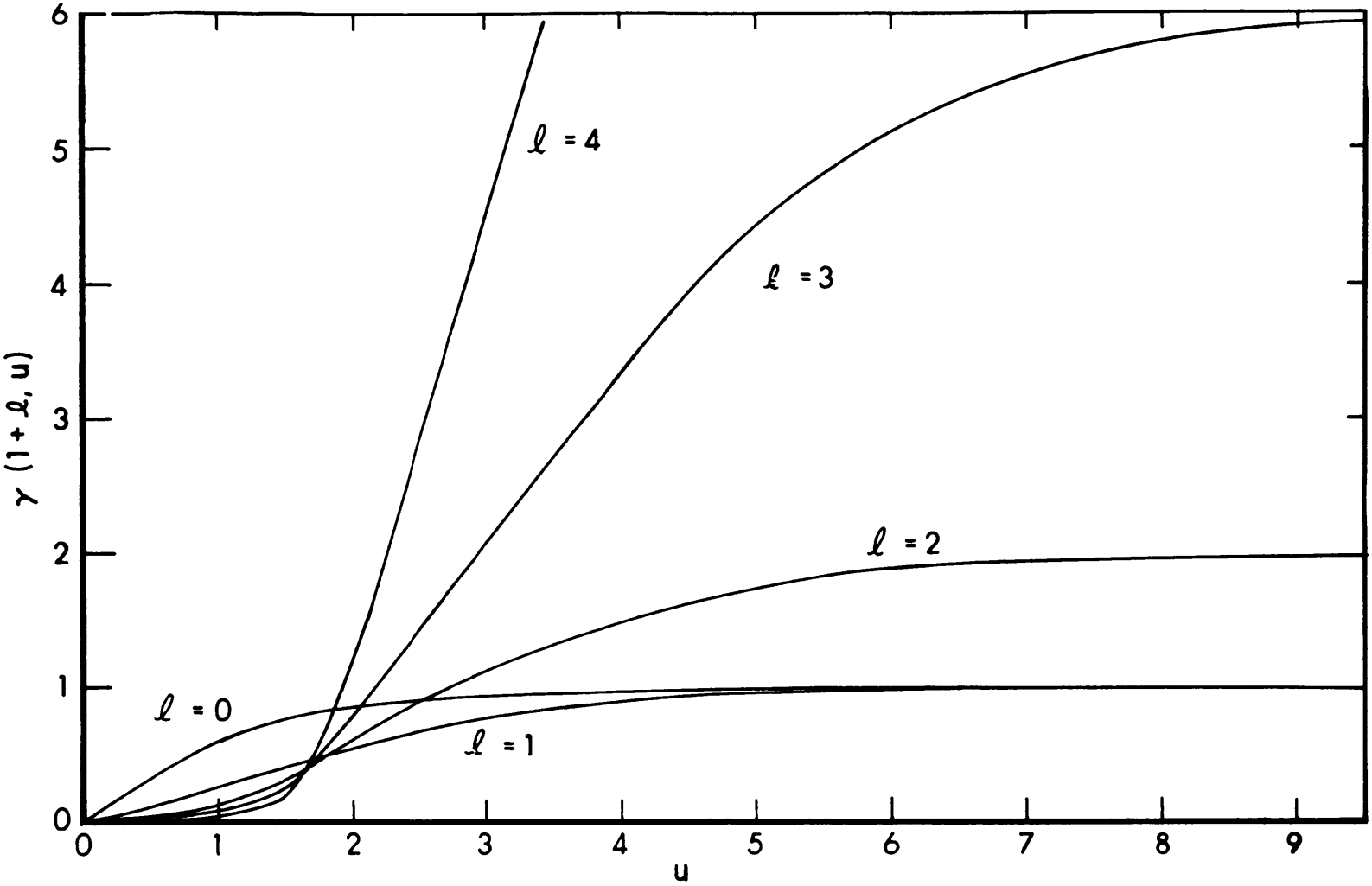

Figure 3. Incomplete Gamma Function.

which has been constructed to simulate much larger populations typical of real warhead natural fragmentation. The advantage of using such a small population is that it is easy to follow in detail various possible procedures. Here we have 28 fragments with total mass M_T = 66.261g and average mass m_{AV} = 66.261/28=2.366g. We can represent this collection graphically by drawing 28 vertical lines of height unity along a horizontal mass axis. These lines will be more closely spaced for smaller m values and become farther apart as m increases. Alternatively we can group the fragments into intervals centered on various mass values and count the number in each group. If we choose equal size intervals each one mass unit wide, we obtain Table II. As we see, the number of fragments, N^E, in each group,

Table II. GROUPING INTO EQUAL SIZE MASS INTERVALS

Interval(g)	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11
N^E	14	4	2	2	1	2	0	1	0	1	1
$N^E(>)$	14	10	8	6	5	3	3	2	2	1	0
$N^E(<)$	14	18	20	22	23	25	25	26	26	27	28

fluctuates erratically for the heavier groups which contain only a few fragments. This is typical of real fragment distributions also. The number of fragments with mass greater than that associated with a given group is $N^E(>)$, while its complement, the number with mass less than that of higher groups, is $N^E(<)$. Of course $N^E(>) + N^E(<) = N_T$, the total number of fragments. Both the cumulative, $N^E(<)$, and complementary cumulative numbers appear somewhat smoother than N^E, so we might expect better agreement if we fit a smooth mathematical function to cumulative numbers rather than to the frequency N^E. We can smooth out the erratic behavior of N by choosing unequal size intervals. For example, the choice in Table III gives much smoother behavior. Of course other choices could introduce erratic behavior

Table III. GROUPING INTO UNEQUAL SIZE INTERVALS

Interval(g)	0-.25	.25-.75	.75-2	2-4	4-6	6-10	10-66.261
N^E	7	6	5	4	3	2	1
$N^E(>)$	21	15	10	6	3	1	0
$N^E(<)$	7	13	18	22	25	27	28

again. For example, choosing 1.95 instead of 2.0 would put 4 fragments in the third group and 5 in the fourth group. This frequently occurs in practice where the choice of mass intervals is made before an experiment is carried out, perhaps for the sake of uniform reporting procedures. Generally speaking, in the literature the mass of each fragment is not reported and only information about pre-chosen groups is given. Consequently, there is no way to choose new groupings which might be more closely represented by certain mathematical functions.

Now let us apply the frequency distribution $dF=d(N/N_T)$ of Equation (20) with $y=m/\rho$ and $\beta=\mu/\rho$ where ρ is the density of the metal case. If we multiply it by the total number of fragments, N_T, and divide by $(1-e^{-u})$ where u is given by Equation (27), we obtain

$$dN = \frac{N_T}{(1-e^{-u})} \quad \frac{1}{\mu\ell}\left(\frac{m-m_o}{\mu}\right)^{\frac{1}{\ell}-1} \exp\left[-\left(\frac{m-m_o}{\mu}\right)^{\frac{1}{\ell}}\right] dm = W(m)\ dm \quad (33)$$

for the number of fragments in the infinitesimal interval from m to m+dm. Note that Equation (33) becomes Equation (1) above for $u\to\infty$, $m_o=0$ and $\ell=3$. The factor $(1-e^{-u})$ insures that

$$\int_{m_o}^{m_u} dN = \left[N_T/(1-e^{-u})\right]\int_{m_o}^{m_u} \exp\left[-\left(\frac{m-m_o}{\mu}\right)^{\frac{1}{\ell}}\right] d\left[\left(\frac{m-m_o}{\mu}\right)^{\frac{1}{\ell}}\right]$$

$$= \left[N_T/(1-e^{-u})\right]\int_o^u e^{-t}\, dt = N_T\ . \quad (34)$$

The cumulative number of fragments is

$$N(<) = \int_{m_o}^{m} dN = N_T\left[1-e^{-t}\right]/\left[1-e^{u}\right], \quad (35)$$

while the complementary cumulative number or number greater than m is the integral from m to m_u (or t to u).

$$N(>) = \int_{m}^{m_u} dN = N_T\left[e^{-t} - e^{-u}\right]/\left[1-e^{-u}\right] \quad (36)$$

so $N(<) + N(>) = N_T$. We can calculate either cumulative number, however, N(>) is the number used to evaluate the lethality of a warhead or the vulnerability of a target. We can determine N(>) experimentally with greater accuracy also, since the number and size of small fragments is difficult to measure. The use of a cutoff mass, m_o, of sufficient size avoids this difficulty, and is also useful because very small fragments are usually not lethal anyway.

In practice, we do not deal with infinitesimals such as dN and dm. Instead we use fragment groupings such that N_i fragments are found in the i^{th} mass group of width dm_i centered on mass m_i. Generally speaking also in fragmentation work the expected number of fragments is large and approximately equal to the expected number of breaks, u. If $u > 10$, then $1-e^{-u} \approx 1$ and may be neglected in Equations (33) through (36). This may not be true for t very close to 0 or u in Equations (35) and (36), but the use of finite groupings prevents this from happening. This is why u can be taken to be effectively infinite, at least in cases of natural fragmentation. For a narrow group of controlled fragments clustered about μ this may not be true for small ℓ, since $u = [(m_u-m_o)/\mu]^{1/\ell}$ with m_u, m_o and μ all of about the size. For finite size intervals Equation (33) becomes

$$N_i = N_T W_i = N_T \left[\left(\frac{\Delta m_i}{\mu \ell}\right) t_i^{(1-\ell)} e^{-t_i} / (1-e^{-u})\right] \qquad (37)$$

where

$$t_i = [(m_i-m_o)/\mu]^{1/\ell} \qquad (38)$$

for the i^{th} mass group. We can use Equation (38) for t_i instead of t in Equations (35) and (36) to take account of finite rather than infinitesimal size mass intervals.

In Equation (37), $W_i = N_i/N_T$ is the calculated probability of finding N_i fragments in the i^{th} group. The probability observed experimentally is $W_i^E = N_i^E / N_T$, where N_i^E is the number found experimentally. We are interested in obtaining functional representations of experimental data, using the probability formulas we have derived. As we have seen, there is no reason to require that ℓ be integer, so we will treat it as an adjustable parameter. In addition, the observed average mass gives us only a rough estimate of $\bar{m}$ (and so μ through Equations (30) to (32)). We will use this as an initial guess for μ and then adjust μ also to represent the data better. The cutoff mass, m_o, will determine the total number of fragments, N_T. For example, in Table I we are assuming that fragments less than m_o = 0.1 g are of no interest in a particular application because they are too small to be lethal. If m_o were 0.2 g instead, then N_T would be 21 instead of 28. If m_o were smaller than 0.1 g then N_T would be larger than 28. However, the smaller we make m_o, the more uncertain

we are about the value of N_T above m_o because of the difficulty of observing very small fragments. The minimum value of m_o is well above molecular mass values, since we are considering a defect method of fragment production. Still the actual number of dust-like fragments emitted after a detonation is undoubtedly very large and experimentally unknowable. Since this number is also of no practical interest for lethality or vulnerability, we will adopt the point of view that N_T is fixed by a choice of finite $m_o > 0$. The upper mass, m_u, can be taken to be the total mass, M_T, which is usually close to the unfragmented case mass. For natural fragmentation this will make u effectively infinite. For controlled fragment groups where m_o, μ and m_u may be close to each other, the choice may be more important as we shall mention later.

If our only interest were to use Equation (36) with u effectively infinite, then we could adopt $\ln N_i(>)$ as our model function, considering it to be a linear function of the variable $(m_i - m_o)^{1/\ell}$ in

$$\ln N_i(>) = \ln N_T - (\mu^{-1/\ell})\ (m_i - m_o)^{1/\ell} \quad . \qquad (39)$$

For fixed ℓ we could adjust μ and so the slope. For fixed μ, an adjustment of ℓ becomes much harder since we are changing the independent variable as well as the slope. Likewise, simultaneous adjustment of ℓ and μ is too laborious to consider. The logarithm of Equation (37) is

$$\ln N_i = \ln [(N_T \Delta m_i)/(\ell \mu^{1/\ell})] + (1-\ell) \ln (m_i - m_o)^{1/\ell} - (\mu^{-1/\ell})\ (m_i - m_o)^{1/\ell} \qquad (40)$$

which is not a linear model function, except for $\ell = 1$. In addition, if u is not effectively infinite, there is no way to make either μ or ℓ appear in a linear fashion in a model function related to Equation (36) or Equation (37).

Here we will use the least squares method to adjust the two parameters $p_1 = \ell$ and $p_2 = \mu$ which appear in the non-linear model functions, Equation (36) and Equation (37). The function to be minimized is the sum of the squared differences between the experimental and calculated values which we will call Sqd (N_g = number of groups):

$$\text{Sqd} = \sum_{i=1}^{N_g} [N_T\ (W_i^E - W_i)]^2 \quad . \qquad (41)$$

We can use our data to obtain initial guesses for the parameters, namely, $p_{1o} = \ell_o$ and $p_{2o} = \mu_o$. If these guesses are reasonable, we can neglect all but linear correction terms in an expansion of W_i about the point (p_{1o}, p_{2o}), so

$$W_i = W_{io} + C_1\left(\frac{d\,W_i}{d\,p_1}\right)_o + C_2\left(\frac{d\,W_i}{d\,p_2}\right)_o \tag{42}$$

where subscript zero means evaluation at the current guess point. Our corrected parameter set ($P_{ko}+C_k$ for k=1,2) becomes the new guess point in an iterative calculation, approaching a best fit as closely as we please. We now put Equation (42) into Equation (41), set the derivatives of Sqd with respect to ℓ and μ equal to zero, divide by $(-2N_T^2)$ and obtain the normal equations. The k^{th} equation is

$$\sum_{i=1}^{N_g}\left[W_i^E - W_{io} - C_1\left(\frac{d\,W_i}{d\,p_1}\right)_o - C_2\left(\frac{d\,W_i}{d\,p_2}\right)_o\right]\left(\frac{d\,W_i}{d\,p_k}\right)_o = 0 \tag{43}$$

where k=1,2... N_p with N_p equal to the number of adjustable parameters (only two here). A rearrangement of Equation (43) gives

$$C_1\left[\sum_{i=1}^{N_g}\left(\frac{d\,W_i}{d\,p_1}\right)_o\left(\frac{d\,W_i}{d\,p_k}\right)_o\right] + C_2\left[\sum_{i=1}^{N_g}\left(\frac{d\,W_i}{d\,p_2}\right)_o\left(\frac{d\,W_i}{d\,p_k}\right)_o\right]$$

$$= \left[\sum_{i=1}^{N_g}\left(W_i^E - W_{io}\right)\left(\frac{d\,W_i}{d\,p_k}\right)_o\right] \tag{44}$$

or, in matrix notation

$$QC = K \tag{45}$$

where

$$Q_{jk} = \sum_{i=1}^{N_g} \left(\frac{d\ W_i}{d\ p_j}\right)_o \left(\frac{d\ W_i}{d\ p_k}\right)_o \tag{46}$$

are the elements of the symmetric matrix Q of dimension N_p x N_p and

$$K_k = \sum_{i=1}^{N_g} \left(W_i^E - W_{io}\right) \left(\frac{d\ W_i}{d\ p_k}\right)_o \tag{47}$$

are the elements of the vector K with the correction vector $C = Q^{-1}K$. Of course in our case with $N_p = 2$, the solution for C is especially simple since Equation (43) consists of only two linear simultaneous equations for the unknowns C_1 and C_2. More elaborate forms of the least squares method could also be used, but this is sufficient for our purpose here.

To carry out this procedure we need derivatives of our model functions which contain factors of the form A^f where $A = (m_i - m_o)/\mu$ and $f = 1/\ell$ or $(\frac{1}{\ell} - 1)$. We recall that the derivative with respect to ℓ can be found by letting

$$Y = \ln\ (A^f) = f \ln A \tag{48}$$

so

$$\frac{dY}{d\ell} = A^{-f}\ \frac{d}{d\ \ell}\ A^f = \ln A \left(\frac{df}{d\ell}\right) \tag{49}$$

and

$$\frac{d}{d\ell}\ (A^f) = A^f \ln A \left(\frac{df}{d\ell}\right) \tag{50}$$

then

$$\frac{d\ W_i}{d\ell} = \left(W_i/\ell\right) \left[-1-(1-t_i) \ln t_i + ue^{-u} \ln u\ /\ (1-e^{-u})\right] \tag{51}$$

$$\frac{d\ W_i}{d\mu} = \left[W_i\ /\ (\mu\ell)\right] \left[-1 + t_i + ue^{-u}\ /\ (1-e^{-u})\right] \tag{52}$$

where W_i is given in Equation (37) and t_i is given in Equation (38).

If we wish to optimize a fit of $N_i(>)$ to experimental data, the function to be minimized is

$$Sqd = \sum_{i=1}^{N_g} \left[N_i^E(>) - N_i(>) \right]^2 \qquad (53)$$

and a new set of normal equations can be found in a similar way. The required derivatives are

$$\frac{d\, N_i(>)}{d\ell} = \left(N_T/\ell\right)\left[t_i e^{-t_i} \ln t_i - \left(\frac{1-e^{-t_i}}{1-e^{-u}}\right) u e^{-u} \ln u \right] \Big/ \left(1-e^{-u}\right) \qquad (54)$$

$$\frac{d\, N_i(>)}{d\mu} = \left[N_T/(\mu\ell)\right]\left[t_i e^{-t_i} - \left(\frac{1-e^{-t_i}}{1-e^{-u}}\right) u e^{-u} \right] \Big/ \left(1-e^{-u}\right) . \qquad (55)$$

A computer program implementing this procedure is given in the Appendix. For cases in which u is effectively infinite, all terms involving u in Equations (51), (52), (54) and (55) vanish as can easily be verified by the use of L' Hospital's rule. Provisions are also made in the appendix for adjusting either ℓ or μ alone, as well as for calculating with fixed ℓ and μ, using either finite m_o and m_u or with $m_o = 0$, $m_u = \infty$ as in Mott's case.

C. Applications

Let us apply our procedure to the sample data of Table I as grouped in Tables II and III. For example, Table IV A compares the number in each group calculated by various procedures with the number found experimentally, N^E. In this Table and Table V m_o was taken to be zero. If we took $m_o = 0.1$ g in Table IV we could obtain slightly better agreement as indicated by a lowering of Sqd from 6.86 to 6.56, accompanied by somewhat changed ℓ and μ. However, this is good enough for our purpose, which is to show that the fitted Sqd is much lower than that for integer ℓ values, namely, 33.53, 24.38 or 61.92. For the fit, m_u was taken to be 66.261 g so that $u = (66.261/.95)^{1/1.39} = 21.2$ which is effectively infinite, giving N approximately equal to zero for the last group. In the Mott-type calculations $m_u = u = \infty$, so N and N(>) are both exactly

zero. This is true because the exponential factors vanish and dominate other factors which increase without limit. Table IV B compares calculated values with experimental values of N(>). A gain from a comparison of Sqd values we see the value of adjusting ℓ and μ. The adjusted $\ell = 2.03$ is almost integer, so the improvement over $\ell = 2$, $\mu = 1.183$ g is not as great as in Part A.

Table V makes the same comparisons as Table IV but for the sample data as grouped in Table III.

Table IV. FRAGMENT DISTRIBUTION FOR SAMPLE IN TABLE II

(Columns marked =1, 2 or 3 use the Mott version.)

	A. N in each group					B. N(>) for each group				
Interval(g)	N^E	N	$\ell = 1$	$\ell = 2$	$\ell = 3$	$N(>)^E$	N(>)	$\ell = 1$	$\ell = 2$	$\ell = 3$
0-1	14	13.51	9.58	9.50	6.84	14	14.96	22.67	14.62	9.48
1-2	4	4.65	6.28	3.41	2.04	10	9.54	14.85	9.08	5.88
2-3	2	2.18	4.11	1.90	1.09	8	7.01	9.73	6.54	4.40
3-4	2	1.14	2.70	1.23	.70	6	5.46	6.38	5.01	3.53
4-5	1	.64	1.77	.86	.49	5	4.41	4.18	3.98	2.95
5-6	2	.38	1.16	.64	.37	3	3.64	2.74	3.24	2.52
6-7	0	.23	.76	.48	.29	3	3.05	1.79	2.69	2.20
7-8	1	.14	.50	.38	.23	2	2.60	1.18	2.26	1.94
8-9	0	.09	.33	.30	.19	2	2.23	.77	1.92	1.73
9-10	1	.06	.21	.25	.16	1	1.94	.51	1.65	1.56
10+	1	.00	0	0	0	0	.14	0	0	0
ℓ	-	1.39	1	2	3	-	2.03	1	2	3
μ(g)	-	.95	2.366	1.183	0.394	-	1.29	2.366	1.183	0.394
Sqd	-	6.86	33.53	24.38	61.92	-	4.46	106.43	6.00	62.00

Table V. FRAGMENT DISTRIBUTION FOR SAMPLE IN TABLE III

(Columns marked ℓ = 1, 2 or 3 used the Mott version.)

	A. N in each group					B. N(>) for each group				
Interval(g)	N^E	N	$\ell = 1$	$\ell = 2$	$\ell = 3$	$N(>)^E$	N(>)	$\ell = 1$	$\ell = 2$	$\ell = 3$
0-.25	7	6.24	2.81	6.58	6.44	21	21.11	26.56	20.23	14.16
.25-.75	6	5.15	4.79	4.75	3.42	15	15.26	22.67	14.62	9.48
.75-2	5	5.44	8.27	4.67	2.82	10	9.69	15.66	9.53	6.14
2-4	4	3.56	6.66	3.02	1.71	6	5.47	7.88	5.70	3.92
4-6	3	1.68	2.86	1.47	.85	3	3.22	3.38	3.58	2.72
6-10	2	1.44	1.61	1.35	.83	1	1.69	.95	2.08	1.83
10+	1	.18	0	0	0	0	.03	0	0	0
ℓ	-	1.72	1	2	3	-	1.81	1	2	3
μ(g)	-	1.41	2.366	1.183	0.394	-	1.23	2.366	1.183	0.394
Sqd	-	4.40	38.01	6.55	23.94	-	.98	125.38	2.56	97.25

As we see the fit gives better agreement than the use of integer ℓ and appropriate μ. If we ignore ℓ=1 (which is never used), only one case (for ℓ=3) gives a worse fit than is found in Table IV. We expect this since the data in Table III was made to appear smoothly decreasing by a choice of mass intervals. For ℓ=1 this choice makes the calculations appear very erratic since a maximum of 8.27 is reached for the third mass group. The probability per unit mass, w(m) in Equation (33), decreases steadily as m increases for any $\ell \geq 1$. However, $W_i = w_i \Delta m_i$ in Equation (37) is the product of w_i with Δm_i so that an increase in Δm_i as in the first intervals of Table V can more than offset a decrease in w_i, resulting in a net increase. This often happens to experimental data too, when we use variable size groupings. If all mass intervals are equal, then the calculated numbers will always decrease as m increases. However, experimentally, erratic behavior can occur even for equal size intervals because of poor statistics. In Table V A with ℓ=1, $N_i = (28/2.366)\, \Delta m_i \exp(-m_i/2.366)$. This gives $N_1 = 2.81$ for $\Delta m_1 = .25$, $m_1 = .125$, etc. as shown. For larger ℓ values this difficulty tends to disappear.

Now let us examine some experimental data obtained from a real munition. In the Netherlands Lindeijer and Liemans[18] published data in the open literature for a 105 mm shell which is presented here as the experimental values in Table VI. An unusual feature of their experiment was the large number of tiny fragments they collected in the sub-gram category compared to the 816 individually weighed and counted (and recorded in Table VI). They were able to collect 2,195 tiny fragments weighing 215 grams altogether, compared to the 816 recorded here which totalled 457 grams. The total number of fragments in Table VI is N_T = 2,228 weighing a total of M_T = 8,561 g, giving m_{AV} = 3.84 g. This population is typical of that reported by other groups in similar experiments. Since the original case mass weighed about 9,000 grams, about 2.5 per cent was lost, presumably as a very large number of very small fragments lost despite the extraordinary efforts of these authors. Since the actual number of small fragments emitted (or created by secondary breakup during recovery) cannot be determined (although it must be extremely large), we see the need for invoking a fixed value of N_T, determined by a non-zero value of m_o.

In Table VI we have divided the Sqd values by the total number of fragments to keep these values reasonably small. From a comparison of these values it is clear that using "best" ℓ and μ can give a better overall representation of experiment than the use of integer ℓ and prescribed $\mu = m_{AV}/(\ell!)$. To avoid crowding in Table VI we have omitted columns for values found using ℓ =1 (worse anyway) and have rounded calculated fragment numbers to integer values, at least when they exceed unity. The Sqd values could have been improved a few more percent in the "best" calculations by using $m_o > 0$, but this is not shown here for the sake of uniformity in comparisons. The low values found for the heavest groups reflect the infinite (or effectively infinite) upper mass limits used. Although this is required in the Mott-Type calculation, improvements in the "best" calculation can also be made by using the mass of the heaviest fragment found instead of the case mass. Note that the "best" ℓ values are not very different from $m_{AV}/(2!)$ = 1.92 g, although the "best" ℓ values are notably lower than ℓ=2, giving better agreement for the more populated groups and worse for those with only a few fragments. When we estimate effectiveness for cases where fragments less than ten grams are important, it is presumably more important to have a better representation of the more populated groups, since there are so few heavy fragments. If these lighter fragments are not important for a particular application, we can always increase m_o, reduce N_T, adjust ℓ and μ and again obtain "best" values for the remaining population. Similar calculations have been carried out for other real munitions (both larger and smaller caliber) and similar results have been obtained.

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18 E. W. Lindeijer and J. S. Leemans, Explosivstoffe 16 (7), 145 (1968).

Instead of using any type of analytical formula, one can of course use the experimental data, especially in this age of the digital computer. However, for some purposes, analytical formulas are desirable. In addition, the smoothing which results from the use of a fitted analytical formula should better represent the average values and average effectiveness of a large number of munitions, the case we are trying to calculate. Use of experimental data with poor statistics giving erratic behavior in heavier mass groups (and certain angular zones) could give poorer effectiveness estimates than use of smooth functions. Presumably the data would become smooth if only we could afford to do enough experiments. Sometimes a few larger fragments might be neglected anyway because they are too slow to be of importance.

Figure 4 compares the curves for N(>) from Table VI. The "best" fit curve is virtually indistinguishable from the experimental curve on the scale of this graph.

Table VI. FRAGMENT DISTRIBUTION FOR THE NATURAL FRAGMENTATION OF A 105 MM SHELL

	A. N in each group				B. N(>) for each group			
Interval(g)	N^E	N	$\ell = 2$	$\ell = 3$	$N(>)^E$	N(>)	$\ell = 2$	$\ell = 3$
0-1	816	804	683	544	1412	1421	1337	895
1-2	464	414	271	175	948	938	921	598
2-3	234	257	162	97	714	689	712	469
3-5	270	282	190	109	444	471	526	360
5-10	233	220	203	118	211	233	309	235
10-15	97	53	89	55	114	104	174	155
15-20	49	15	47	32	65	53	109	113
20-25	28	5	28	21	37	29	73	87
25-30	13	2	17	15	24	17	51	70
30-35	5	.51	12	11	19	10	36	57
35-40	1	.18	8	8	18	6	27	48
40-45	4	.07	6	6	14	4	20	40
45-50	4	.03	4	5	10	3	15	35
50+	10	.00	0	0	0	.00	0	0
ℓ	-	1.27	2	3	-	1.68	2	3
μ	-	2.18	1.92	.64	-	1.91	1.922	.64
Sqd/N_T	-	3.33	30.38	97.67	-	1.25	13.69	210.60

Finally, let us consider an application to controlled fragmentation. It is obvious that for particular targets or target classes there will be an optimum fragment size (also dependent somewhat on engagement conditions). Smaller fragments are wasted since they cannot sufficiently damage the target, while fragments which are too large will kill the target but inefficiently since they will of necessity be fewer in number, so less likely to hit without special aiming devices. The ultimate in control is the pre-formed fragment which is often used when launch stresses are relatively low as for missile warheads. However, gun-launched warheads require strong casings which do not permit weakening the shell by grooving. For such shells other techniques have been devised which permit undiminished case integrity coupled with some degree of fragment size control. Examples are the use of electron beam scoring, selective carburization of the steel and grooving of the explosive at the shell wall (with or without a liner). At BRL Meissner and Kineke[19] have compared various techniques with each other and with natural fragmentation. For example, they used a cylindrical steel case 10.16 cm long with outer diameter 7.6 cm and wall thickness 0.24 cm, filled with Composition B explosive to make such comparisons. Table VII compares the natural fragmentation they observed with values calculated by various methods. Clearly a better representation can be obtained by using best values of ℓ and μ. Since the mass of the largest fragment was recorded and is known to be less than 14 g, this value was used for m_u.

In other experiments Meissner and Kineke have grooved the case or the explosive in an effort to control fragment size. In particular they used 22 longitudinal grooves evenly spaced together with 2 circumferential grooves, each one 1.27 cm from either end. For the sake of illustration we will discuss the results they obtained using a grooved explosive. As we see from Figure 5 and Table VIII, they succeeded in producing 41 fragments with mass greater than 13 g, compared to only one such fragment when no effort was made to control fragment size (Table VII). These rod-like fragments centered about a mass near 17 g. In addition, they produced about 21 end fragments in a group which centered near 6 or 7 g. The rest of the case mass seems to be in a naturally fragmented group with most of the mass below 5 g. Column one of Table VIII (N^E) gives the number observed in each mass interval. This column has been divided into the three groups in the next three columns, using Figure 5 as a guide. Here N_1^E is the natural part of the fragmentation while N_2^E and N_3^E are the partially

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19 R. Meissner and J. Kineke, report to be published.

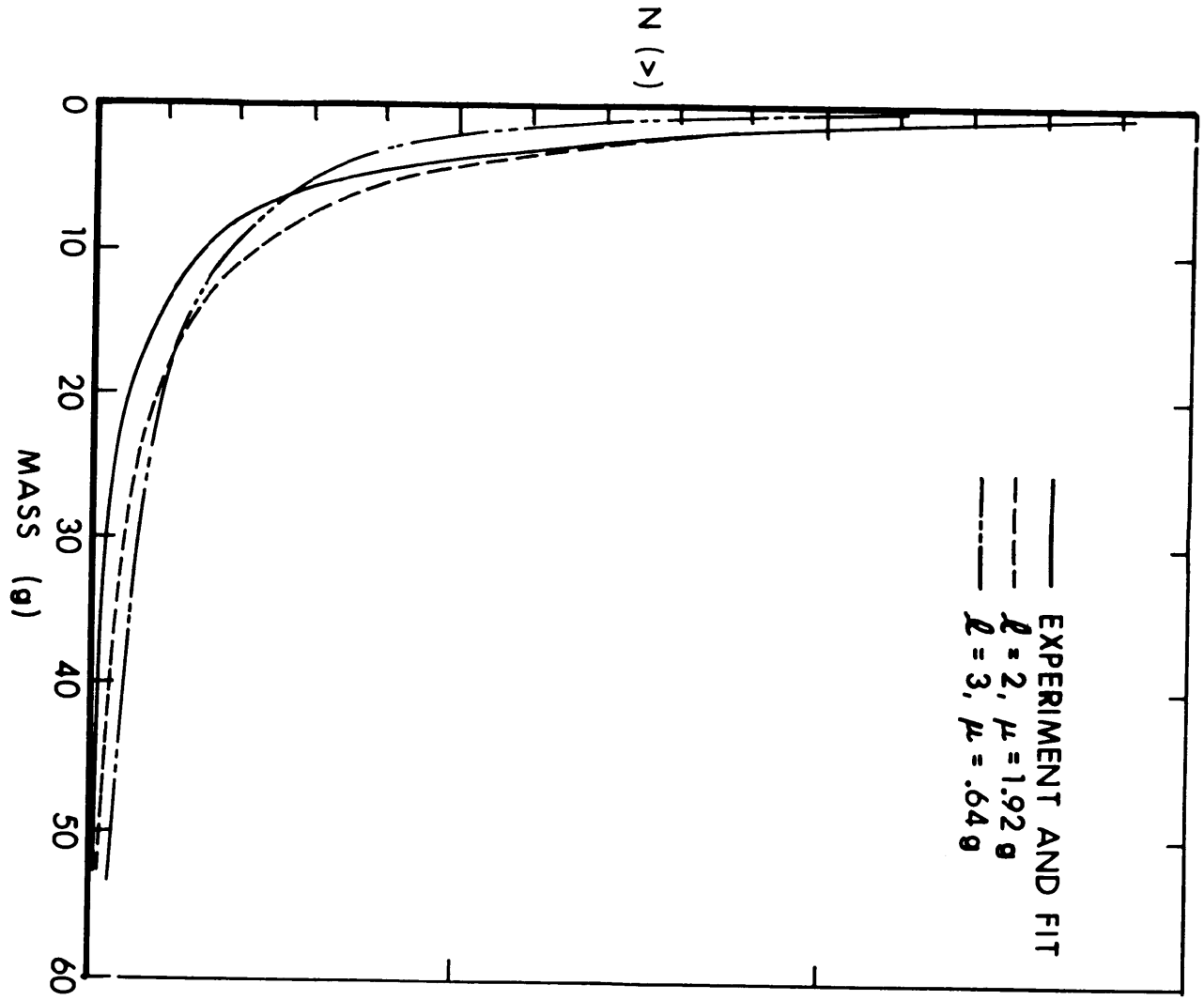

Figure 4. Number of fragments with mass greater than a given mass (105 mm shell).

Table VII. FRAGMENT DISTRIBUTION FOR THE NATURAL FRAGMENTATION OF A CYLINDER

	A. N in each group				B. N(>) for each group			
Interval(g)	N^E	N	$\ell = 2$	$\ell = 3$	$N(>)^E$	N(>)	$\ell = 2$	$\ell = 3$
0-1	216	205	156	110	235	252	226	146
1-2	65	81	54	32	170	154	136	89
2-3	47	42	30	17	123	107	96	66
3-4	44	24	19	11	79	78	73	52
4-5	20	14	13	8	59	58	57	43
5-6	17	9	10	6	42	44	46	37
6-7	21	6	7	4	21	34	37	32
7-8	5	4	6	3	16	25	31	28
8-9	8	3	4	3	8	19	26	25
9-10	3	2	4	2	5	14	22	22
10-11	0	1	3	2	5	10	19	20
11-12	3	.83	2	1.7	2	6	16	18
12-13	1	.59	2	1.5	1	3	14	17
13-m_u	1	.42	0	0	0	1	0	0
ℓ	-	1.35	2	3	-	1.83	2	3
μ	-	1.24	1.05	.35	-	1.46	1.05	.35
Sqd/N_T	-	2.61	10.97	33.21	-	2.91	8.17	45.06

Note: $m_u = 14g$ for best calculations m_u = infinity for $\ell = 2$ or 3.

controlled parts. Since the controlled parts exhibit maxima, we must use $\ell<1$ in Equation (37) to represent them. This may be seen by examining the mass-dependent part of Equation (37),

$$w = \frac{1}{\mu\ell}\left(\frac{m - m_o}{\mu}\right)^{\frac{1}{\ell}-1} \exp\left[-\left(\frac{m - m_o}{\mu}\right)^{\frac{1}{\ell}}\right] \qquad (56)$$

where $\frac{1}{\mu^{\ell}}$ has been included to make it equal to w in Equation (33)° For $\ell>1$, w decreases smoothly from infinity at $m = m_o$ to zero as m becomes large, as we have seen in our representations of natural fragmentation. For $\ell=1$ we have a simple exponential, finite at $m = m_o$. For $0<\ell<1$, w vanishes for $m = m_o$ or infinity, but has a maximum in between.

In Table VIII the columns marked N_1, N_2 and N_3 give the fragment

TABLE VIII. PARTIALLY CONTROLLED FRAGMENTATION OF A CYLINDER

(blanks mean zero)

Interval	N^E	$N_1{}^E$	$N_2{}^E$	$N_3{}^E$	N	N_1	N_2	N_3	$N(>)^E$	N(>)
0-1(g)	63	63			61.22	61.22			94	89.04
1-2	15	15			18.33	18.33			79	70.71
2-3	13	13			5.78	5.78			66	64.93
3-4	3	3			1.87	1.87			63	63.06
4-5	6	1	5		4.32	.61	3.71		57	58.74
5-6	5		5		7.00		7.00		52	51.79
6-7	8		8		5.86		5.86		44	45.88
7-8	3		3		3.20		3.20		41	42.68
8-9					1.24		1.24		41	41.44
9-10					.36		.36		41	41.08
10-11									41	41.08
11-12									41	41.08
12-13									41	41.08
13-14	1			1	1.97			1.97	40	39.11
14-15	6			6	4.61			4.61	34	34.50
15-16	4			4	6.09			6.09	30	28.41
16-17	10			10	6.48			6.48	20	21.93
17-18	3			3	6.03			6.03	17	15.90
18-19	5			5	5.06			5.06	12	10.84
19-20	4			4	3.89			3.89	8	6.95
20-21	5			5	2.77			2.77	3	4.18
21-22	1			1	1.83			1.83	2	2.35
22-23	0			0	1.14			1.14	2	1.21
23-24	1			1	.66			.66	1	.55
24-25	0			0	.36			.36	1	.19
25-26	1			1	.19			.19	0	0

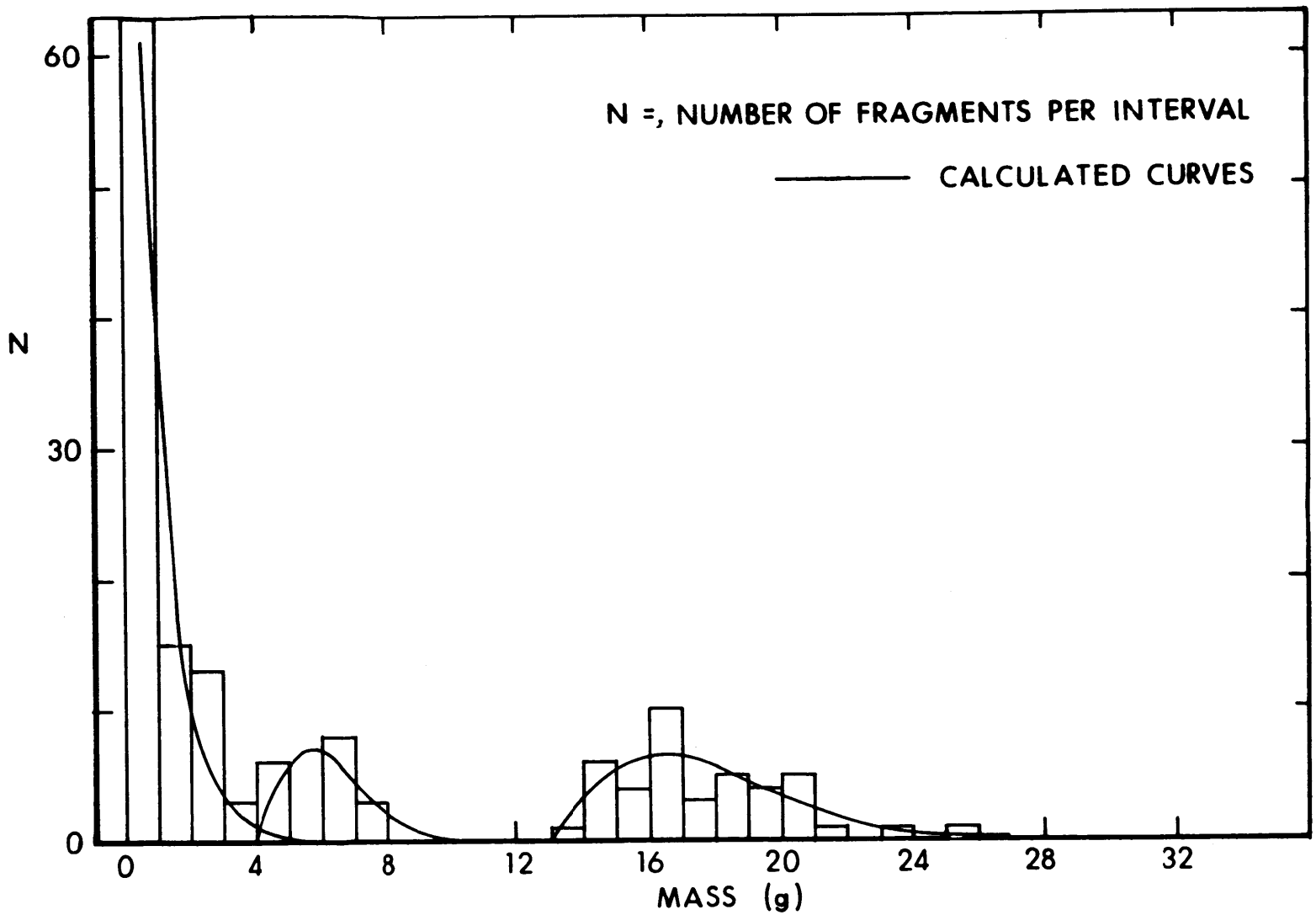

Figure 5. Partially Controlled Fragmentation of a Cylinder.

numbers calculated using adjusted ℓ and μ values. For N_1 the total number of fragments was taken to be 95 (as in N_1^E) while $m_o=0$, giving $\ell=1.04$ $\mu=.82$ g and $Sqd/N_{T1}=.77$. For N_2 the total number $N_{T2}=21$ (as in N_2^E). In addition, the values $m_o=4$ g and $m_\mu=10$ g were used, giving $\ell=.53$, $\mu=2.47$ g and $Sqd/T_{T2}=.56$. Here the sum of m_o and μ gives an indication of the peak position. Using $m_o>0$ seems to give a better representation than $m_o=0$ here. For N_3, $N_{T3}=41$ and $m_o=13$ g, giving $\ell=.54$, $\mu=5.18$ and $Sqd/N_{T3}=.89$. The calculated values for these groups are shown by the solid lines in Figure 5. The agreement appears to be reasonable. The last two columns in Table VIII are the complementary cumulative distributions obtained by subtraction, starting with the total number observed, $N_T^E=157$, or with the total number calculated $N_T=150.26$. Again the agreement is reasonably good.

Strictly speaking, we should carry out a simultaneous fit of three theoretical curves to the data, using m_o and N_T as well as ℓ and μ for each curve as adjustable parameters. This more complicated procedure (or one similar to it) is actually used in certain types of spectroscopy where the statistics are extremely good in order to extract information about partially resolved lines. However, in the present case where the statistics are so poor, the procedure used seems adequate.

III. SUMMARY

We have seen that Mott's distribution is a special case of more general probability distributions which are widely used to describe phenomena in various fields. In spite of this it can be given a rational basis in terms suited to fragmentation, namely, the random distribution of defects in solids. Methods of grouping fragment populations have been discussed, and a general probability distribution has been applied to several collections of experimental data. In particular, it has been shown that this approach can be used not only for natural fragmentation, but also for controlled fragmentation.

APPENDIX

This computer code for the Hewlett-Packard 9845B offers a number of options in fitting Equations (36) and (37) of the text to experimental fragment size data. These should be self-evident from the various questions asked by the interactive display statements which occur throughout the program. Since the first part of the program aims to optimize a fit to the number in each group, the output subroutine (lines 1470 to 1730) always prints N values but permits N(>) values to be printed if desired, even though these have not been optimized. Similarly, in the output subroutine for part two (lines 2970 to 3220), N(>) is always printed, but N is an option. This is particularly useful when the program is used without adjusting the parameters. This occurs in two sections, one starting on line 1030 in which the values of m_o and m_u specified at input are used, the other starting on line 2460 in which $m_o=0$ and $m_u=\infty$.

```
10 REM FRAG.   FIT TWO-PARAMETER FUNCTIONS TO DATA. FIRST PART FITS NUMBER VS
       MASS GROUP. SECOND PART FITS COMPLEMENTARY CUMULATIVE NUMBER.
20 REM INPUT: MMIN=CUTOFF MASS,NTOT=TOTAL NUMBER OF FRAGMENTS,NG=NUMBER OF MASS
       GROUPS,MUP(I)=UPPER LIMIT OF ITH GROUP,N(I)=NUMBER IN GROUP I.
30 REM P1=L AND P2=MU ARE ADJUSTABLE PARAMETERS.
40     OPTION BASE 1
50     PRINTER IS 0
60     I1=0
70     DIM M(50),W(50),D(50),Dev(50),Mup(50),N(50),Dm(50),Ngmc(50),Ngme(50),V(50)
80     DIM Dv(50)
90     IMAGE 2D,5(8D.2D)
100    DISP "MMIN,NTOT,NG";
110    INPUT Mmin,Ntot,Ng
120    PRINT "MMIN=";Mmin;"NTOT=";Ntot;"NG=";Ng
130    PRINT " I      MUP(I)       N(I)      NGME(I)"
140    FOR I=1 TO Ng
150    DISP "MUP(I),N(I)";
160    INPUT Mup(I),N(I)
170    IF I>1 THEN 230
180    Dm(1)=Mup(1)-Mmin
190    M(1)=(Mup(1)+Mmin)/2
200    D(1)=N(1)/Ntot
210    Ngme(1)=Ntot-N(1)
220    GOTO 270
230    Dm(I)=Mup(I)-Mup(I-1)
240    M(I)=(Mup(I)+Mup(I-1))/2
250    D(I)=N(I)/Ntot
260    Ngme(I)=Ngme(I-1)-N(I)
270    PRINT USING 90;I,Mup(I),N(I),Ngme(I)
280    NEXT I
290    DISP "FIT FRAGMENT NUMBER IN EACH GROUP? (0 IF NO, 1 IF YES)";
300    INPUT Ind
310    IF Ind=0 THEN GOTO 1740
320    PRINT "FIT EXPRESSION FOR NUMBER IN ITH GROUP TO DATA"
330    PRINT
340    DISP "P1=? P2=? 0,0 MEANS SKIP THIS PART ";
350    INPUT P1,P2
360    IF P1=0 THEN GOTO 730
370    PRINT "ADJUST P1=";P1;" AND P2=";P2
380    Q11=Q12=Q22=K1=K2=Sqd=0
390    FOR I=1 TO Ng
400    A=(M(I)-Mmin)/P2
410    X=A^(1/P1)
420    B=(Mup(Ng)-Mmin)/P2
430    U=B^(1/P1)
440    T1=EXP(-X)
450    T2=EXP(-U)
460    T3=U*T2/(1-T2)
470    W(I)=Dm(I)*X*T1/(P1*P2*A*(1-T2))
480    Ngmc(I)=Ntot*(T1-T2)/(1-T2)
490    D1=W(I)*(-1-LOG(X)*(1-X)+T3*LOG(U))/P1
500    D2=W(I)*(-1+X+T3)/(P1*P2)
510    Q11=Q11+D1^2
520    Q12=Q12+D1*D2
530    Q22=Q22+D2^2
540    Dev(I)=D(I)-W(I)
550    K1=K1+D1*Dev(I)
560    K2=K2+D2*Dev(I)
570    Sqd=Sqd+(Dev(I)*Ntot)^2
```

```
580   NEXT I
590   PRINT "SQD=";Sqd
600   Den=Q11*Q22-Q12^2
610   C1=(K1*Q22-K2*Q12)/Den
620   C2=(K2*Q11-K1*Q12)/Den
630   Rm=MAX(ABS(C1/P1),ABS(C2/P2))
640   IF Rm<=.2 THEN 670
650   C1=.2/Rm*C1
660   C2=.2/Rm*C2
670   IF (ABS(C1)<.001) AND (ABS(C2)<.001) THEN 720
680   P1=P1+C1
690   P2=P2+C2
700   PRINT "P1= ";P1;"P2= ";P2
710   GOTO 380
720   GOSUB Output1
730   DISP "P1=? P2=? ADJUST P1 ONLY. 0,0 TO SKIP";
740   INPUT P1,M2
750   IF P1=0 THEN GOTO 1030
760   PRINT
770   PRINT "      WITH P2=";M2;", ONLY ADJUST P1=";P1
780   Q11=K1=Sqd=0
790   FOR I=1 TO Ng
800   A=(M(I)-Mmin)/M2
810   X=A^(1/P1)
820   U=((Mup(Ng)-Mmin)/M2)^(1/P1)
830   T2=EXP(-U)
840   T3=U*T2/(1-T2)
850   W(I)=Dm(I)*X*EXP(-X)/(P1*M2*A*(1-T2))
860   Ngmc(I)=Ntot*(EXP(-X)-T2)/(1-T2)
870   D1=W(I)*(-1-LOG(X)*(1-X)+T3*LOG(U))/P1
880   Q11=Q11+D1^2
890   Dev(I)=D(I)-W(I)
900   Sqd=Sqd+(Dev(I)*Ntot)^2
910   K1=K1+D1*Dev(I)
920   NEXT I
930   PRINT "SQD=";Sqd
940   C1=K1/Q11
950   Rm=ABS(C1/P1)
960   IF Rm<=.2 THEN 980
970   C1=.2/Rm*C1
980   IF ABS(C1)<.001 THEN 1020
990   P1=P1+C1
1000  PRINT "P1=";P1
1010  GOTO 780
1020  GOSUB Output1
1030  DISP "P1=? P2=? BOTH KEPT CONSTANT. 0,0 TO SKIP";
1040  INPUT M1,M2
1050  IF M1=0 THEN GOTO 1170
1060  Sqd=0
1070  PRINT "    USE CONSTANT P1=";M1;" AND P2=";M2
1080  FOR I=1 TO Ng
1090  X=((M(I)-Mmin)/M2)^(1/M1)
1100  U=((Mup(Ng)-Mmin)/M2)^(1/M1)
1110  T2=EXP(-U)
1120  W(I)=Dm(I)*X*EXP(-X)/(M1*(M(I)-Mmin)*(1-T2))
```

```
1130  Ngmc(I)=Ntot*(EXP(-X)-T2)/(1-T2)
1140  Dev(I)=D(I)-W(I)
1150  NEXT I
1160  GOSUB Output1
1170  DISP "P1=? P2=? ADJUST P2 ONLY. 0,0 TO SKIP";
1180  INPUT M1,P2
1190  IF M1=0 THEN 1450
1200  PRINT "   WITH P1=";M1;",ONLY ADJUST P2=";P2
1210  Q22=K2=Sqd=0
1220  FOR I=1 TO Ng
1230  X=((M(I)-Mmin)/P2)^(1/M1)
1240  U=((Mup(Ng)-Mmin)/P2)^(1/M1)
1250  T2=EXP(-U)
1260  T3=U*T2/(1-T2)
1270  W(I)=Dm(I)*X*EXP(-X)/(M1*(M(I)-Mmin)*(1-T2))
1280  Ngmc(I)=Ntot*(EXP(-X)-T2)/(1-T2)
1290  D2=W(I)*(-1+X+T3)/(M1*P2)
1300  Q22=Q22+D2^2
1310  Dev(I)=D(I)-W(I)
1320  Sqd=Sqd+(Dev(I)*Ntot)^2
1330  K2=K2+D2*Dev(I)
1340  NEXT I
1350  PRINT "SQD=";Sqd
1360  C2=K2/Q22
1370  Rm=ABS(C2/P2)
1380  IF Rm<=.2 THEN 1400
1390  C2=.2/Rm*C2
1400  IF ABS(C2)<.001 THEN 1440
1410  P2=P2+C2
1420  PRINT "P2=";P2
1430  GOTO 1210
1440  GOSUB Output1
1450  GOTO 1740
1460  REM END OF PART ONE
1470 Output1: Sqd=0
1480  PRINT
1490 PRINT "   **NUMBER IN ITH GROUP**   "
1500  PRINT " I       M(I)      EXPER     CALC        DEVIATION"
1510  FOR I=1 TO Ng
1520  Tw=Ntot*W(I)
1530  Td=Ntot*D(I)
1540  Tdev=Ntot*Dev(I)
1550  Sqd=Sqd+Tdev^2
1560  PRINT USING 90;I,M(I),Td,Tw,Tdev
1570  NEXT I
1580  PRINT "SQD=";Sqd
1590  DISP "PRINT NUMBER WITH MASS GREATER THAN M ? (0,1)";
1600  INPUT Ind
1610  IF Ind=0 THEN GOTO 1720
1620  Sqd=0
1630  PRINT
1640  PRINT "  **NUMBER WITH MASS GREATER THAN M**"
1650  PRINT " I      M(I)       EXPER       CALC       DEVIATION"
```

```
1660  FOR I=1 TO Ng
1670  Dev=Ngmc(I)-Ngme(I)
1680  Sqd=Sqd+Dev^2
1690  PRINT USING 90;I,M(I),Ngme(I),Ngmc(I),Dev
1700  NEXT I
1710  PRINT "SQD=";Sqd
1720  PRINT
1730  RETURN
1740  PRINT
1750  PRINT "FIT EXPRESSION FOR NUMBER WITH MASS GREATER THAN M(I) TO DATA "
1760  PRINT
1770  DISP "P1=? P2=? (0,0 MEANS SKIP THIS PART)";
1780  INPUT P1,P2
1790  IF P1=0 THEN GOTO 2160
1800  PRINT "        ADJUSTING P1=";P1;" AND P2=";P2
1810  Q11=Q12=Q22=K1=K2=Sqd=0
1820  FOR I=1 TO Ng
1830  A=(M(I)-Mmin)/P2
1840  X=A^(1/P1)
1850  U=((Mup(Ng)-Mmin)/P2)^(1/P1)
1860  T1=EXP(-X)
1870  T2=EXP(-U)
1880  T3=(1-T1)/(1-T2)
1890  W(I)=Dm(I)*X*T1/(P1*P2*A*(1-T2))
1900  Ngmc(I)=Ntot*(T1-T2)/(1-T2)
1910  D1=Ntot*(X*T1*LOG(X)-T3*U*T2*LOG(U))/P1/(1-T2)
1920  D2=Ntot*(X*T1-T3*U*T2)/P1/P2/(1-T2)
1930  Q11=Q11+D1^2
1940  Q12=Q12+D1*D2
1950  Q22=Q22+D2^2
1960  Dev(I)=Ngme(I)-Ngmc(I)
1970  Sqd=Sqd+Dev(I)^2
1980  K1=K1+D1*Dev(I)
1990  K2=K2+D2*Dev(I)
2000  NEXT I
2010  PRINT "SQD=";Sqd
2020  Den=Q11*Q22-Q12^2
2030  C1=(K1*Q22-K2*Q12)/Den
2040  C2=(K2*Q11-K1*Q12)/Den
2050  REM LIMIT SIZE OF CORRECTIONS
2060  Rm=MAX(ABS(C1/P1),ABS(C2/P2))
2070  IF Rm<=.2 THEN 2100
2080  C1=.2/Rm*C1
2090  C2=.2/Rm*C2
2100  IF (ABS(C1)<.001) AND (ABS(C2)<.001) THEN 2150
2110  P1=P1+C1
2120  P2=P2+C2
2130  PRINT "P1= ";P1;"P2= ";P2
2140  GOTO 1810
2150  GOSUB Output2
2160  DISP "P1=? P2=? ADJUST P1 ONLY. 0,0 MEANS SKIP";
2170  INPUT P1,M2
2180  IF P1=0 THEN GOTO 2460
2190  PRINT "            ADJUST P1=";P1;" ONLY WITH P2=";M2
2200  Q11=K1=Sqd=0
2210  FOR I=1 TO Ng
2220  A=(M(I)-Mmin)/M2
2230  X=A^(1/P1)
```

```
2240  U=((Mup(Ng)-Mmin)/M2)^(1/P1)
2250  T1=EXP(-X)
2260  T2=EXP(-U)
2270  T3=(1-T1)/(1-T2)
2280  W(I)=Dm(I)*X*T1/(P1*M2*A*(1-T2))
2290  Ngmc(I)=Ntot*(T1-T2)/(1-T2)
2300  D1=Ntot*(X*T1*LOG(X)-T3*U*T2*LOG(U))/P1/(1-T2)
2310  Q11=Q11+D1^2
2320  Dev(I)=Ngme(I)-Ngmc(I)
2330  Sqd=Sqd+Dev(I)^2
2340  K1=K1+D1*Dev(I)
2350  NEXT I
2360  PRINT "SQD=";Sqd
2370  C1=K1/Q11
2380  Rm=ABS(C1/P1)
2390  IF ABS(Rm)<=.2 THEN 2410
2400  C1=.2/Rm*C1
2410  IF ABS(C1)<.001 THEN 2450
2420  P1=P1+C1
2430  PRINT "P1=";P1
2440  GOTO 2200
2450  GOSUB Output2
2460  DISP "P1=? P2=? BOTH CONSTANT, USING MOTT FORMULAS.  0,0 MEANS SKIP";
2470  INPUT M1,M2
2480  IF M1=0 THEN GOTO 2660
2490  Sqd=0
2500  PRINT "    USE CONSTANT P1=";M1;" AND  P2=";M2;"IN MOTT FORMULAS"
2510  V(1)=Mup(1)/2
2520  Dv(1)=Mup(1)
2530  FOR I=2 TO Ng
2540  V(I)=M(I)
2550  Dv(I)=Dm(I)
2560  NEXT I
2570  FOR I=1 TO Ng
2580  X=(V(I)/M2)^(1/M1)
2590  T1=EXP(-X)
2600  W(I)=Dv(I)*X*T1/(M1*V(I))
2610  Ngmc(I)=Ntot*T1
2620  IF I<Ng THEN 2640
2630  Ngmc(I)=W(I)=0
2640  NEXT I
2650  GOSUB Output2
2660  DISP "P1=? P2=? ADJUST P2 ONLY. 0,0 MEANS SKIP";
2670  INPUT M1,P2
2680  IF M1=0 THEN GOTO 2950
2690  PRINT "ADJUST P2=";P2;" ONLY, WITH CONSTANT P1=";M1
2700  Q22=K2=Sqd=0
2710  FOR I=1 TO Ng
2720  X=((M(I)-Mmin)/P2)^(1/M1)
2730  U=((Mup(Ng)-Mmin)/P2)^(1/M1)
2740  T1=EXP(-X)
2750  T2=EXP(-U)
2760  T3=(1-T1)/(1-T2)
2770  W(I)=Dm(I)*X*T1/(M1*(M(I)-Mmin)*(1-T2))
2780  Ngmc(I)=Ntot*(T1-T2)/(1-T2)
```

```
2790  D2=Ntot*(X*T1-T3*U*T2)/M1/P2/(1-T2)
2800  Q22=Q22+D2^2
2810  Dev(I)=Ngme(I)-Ngmc(I)
2820  Sqd=Sqd+Dev(I)^2
2830  K2=K2+Dev(I)*D2
2840  NEXT I
2850  PRINT "SQD=";Sqd
2860  C2=K2/Q22
2870  Rm=ABS(C2/P2)
2880  IF Rm<=.2 THEN 2900
2890  C2=.2/Rm*C2
2900  IF ABS(C2)<.001 THEN 2940
2910  P2=P2+C2
2920  PRINT "P2=";P2
2930  GOTO 2700
2940  GOSUB Output2
2950  STOP
2960  END
2970 Output2: Sqd=0
2980  PRINT
2990  PRINT "  **NUMBER WITH MASS GREATER THAN M**"
3000  PRINT " I       M(I)       EXPER      CALC         DEVIATION"
3010  FOR I=1 TO Ng
3020  Dev=Ngmc(I)-Ngme(I)
3030  Sqd=Sqd+Dev^2
3040  PRINT USING 90;I,M(I),Ngme(I),Ngmc(I),Dev
3050  NEXT I
3060  PRINT "SQD=";Sqd
3070  DISP "PRINT NUMBER IN ITH GROUP? (0,1)";
3080  INPUT Ind
3090  IF Ind=0 THEN GOTO 3210
3100  PRINT
3110 PRINT "   **NUMBER IN ITH GROUP**   "
3120  PRINT " I       M(I)       EXPER       CALC       DEVIATION"
3130  Sqd=0
3140  FOR I=1 TO Ng
3150  Nc=W(I)*Ntot
3160  Dev=Nc-N(I)
3170  Sqd=Sqd+Dev^2
3180  PRINT USING 90;I,M(I),N(I),Nc,Dev
3190  NEXT I
3200  PRINT "SQD=";Sqd
3210  PRINT
3220  RETURN
```

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